WACH MODULES AND IND-ALGEBRAICITY OF THE EMERTON–GEE STACK

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1. Introduction

Let us recall the basic definitions from the previous talks: Let Nilp denote the category of $\mathbb{Z}_p$-algebras where p is nilpotent and let $\mathrm{Nilp}^{\mathrm{ft}}$ denote the full subcategory consisting of finitely generated $\mathbb{Z}_p$-algebras where p is nilpotent. We also consider the variants Nilp_a and $\mathrm{Nilp}_a^{\mathrm{ft}}$ consisting of those objects where p^a is zero for fixed $a \in \mathbb{Z}_{\geq 1}$.

Recall that for A an object of $\mathrm{Nilp}^{\mathrm{ft}}$ we have topological rings

$$\begin{aligned} \mathbb{A}_A^+ &= A[[T]] \\ \mathbb{A}_A &= A[[T]][T^{-1}], \end{aligned}$$

where the topology on $\mathbb{A}_A$ is such that $\mathbb{A}_A^+$ is an open subring, equipped with the T-adic topology. Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$ and let $\epsilon : \Gamma \to \mathbb{Z}_p^\times$ be the cyclotomic character. We equip our topological rings with an action of Γ determined by

$$\gamma(1+T) = (1+T)^{\epsilon(\gamma)},$$

and an action of Frobenius determined by $\varphi(1+T) = (1+T)^p$. Recall the following definitions for objects A of $\mathrm{Nilp}^{\mathrm{ft}}$:

Definition 1.0.1. A rank d étale φ-module over A is a rank d projective $\mathbb{A}_A$-module M equipped with a continuous semilinear action of φ, such that M is generated by the image of $\varphi_M : M \to M$.

Definition 1.0.2. A rank d étale (φ, Γ)-module over A is a rank d projective $\mathbb{A}_A$-module M equipped with commuting (continuous) semilinear actions of φ and Γ, such that M is generated by the image of $\varphi_M : M \to M$.

We will consider groupoid valued functors

- $\mathcal{R}_d$ which sends $A \in \mathrm{Nilp}^{\mathrm{ft}}$ to the stack of rank d étale φ-modules over A and the substack $\mathcal{R}_{d,\mathrm{free}} \subset \mathcal{R}_d$ consisting of those étale φ-modules over A whose underlying module is free fpqc locally on A.
- $\mathcal{R}_d^a$ which is the restriction of $\mathcal{R}_d$ to $\mathrm{Nilp}_a^{\mathrm{ft}}$ and similarly $\mathcal{R}_{d,\mathrm{free}}^a$.
- $\mathcal{X}_d$ sending $A \in \mathrm{Nilp}$ to the groupoid of rank d étale (φ, Γ)-module over A.

1.1. **Descent.** It is not obvious that any of the groupoid valued functors we have written down are stacks in the fpqc topology. For this we consider the following result of Drinfeld:

Proposition 1.1.1. *The following groupoid valued functors are stacks in the fpqc topology (and the same results hold with $\mathbb{A}_A$ replaced by $\mathbb{A}_A^+$)*

- *The stack of finitely generated projective $\mathbb{A}_A$ -modules*
- *The stack of projective $\mathbb{A}_A$-modules of rank d*
- *The stack of finitely generated $\mathbb{A}_A$-modules which are fpqc locally free of rank d*

Remark 1.1.2. These results are straightforward to prove for $\mathbb{A}_A^+$ if we restrict to Noetherian rings A and finitely presented faithfully flat maps $A \to B$. Indeed in that case the induced maps

$$\mathbb{A}_A^+/T^n \to \mathbb{A}_B^+/T^n$$

are faithfully flat for all n, and so we can apply usual faithfully flat descent results to M/T^n for all n and in fact to the compatible system.

The functor $\mathcal{R}_d^a$ is just[1] the stack of φ-equivariant objects of the stack of projective $\mathbb{A}_A$-modules of rank d, and is therefore a stack in the fqpc topology. Showing that $\mathcal{X}_d$ satisfies fpqc descent is slightly more subtle, and we will discuss it later.

2. Representability results for stacks of φ-modules

The main representability result that we will discuss in this section is the following result of Emerton–Gee

Theorem 2.0.1 (Theorem 1.2.1 of [3])**.** *The stack $\mathcal{R}_d^a$ can be written as a (countably indexed) inductive limit of finite-type algebraic stacks over $\mathbb{Z}/p^a\mathbb{Z}$ along closed immersions.*

Roughly speaking, we will write it as the union over the 'substack' consisting of those étale φ-modules admitting a $\mathbb{A}_A^+$-lattice where the relative position of Frobenius is 'bounded'.

2.1. **Bounded objects.** Let $F \in \mathbb{Z}_p[T]$ be a polynomial that is congruent to a positive power of T modulo p. Let $h \in \mathbb{Z}_{\geq 1}$ be an integer, then for $A \in \mathrm{Nilp}^{\mathrm{ft}}$ we define:

Definition 2.1.1. A *φ-module of F-height at most h* over $\mathbb{A}_A^+$ is a a finitely generated T-torsion free $\mathbb{A}_A^+$-module $\mathscr{M}$ equipped with a φ-semilinear map $\varphi_{\mathscr{M}}$ whose linearisation

$$\Phi_{\mathscr{M}} : \varphi^* \mathscr{M} \to \mathscr{M}$$

[1] This is not actually true, because of continuity. I don't know why the the continuity of φ_M cuts out an algebraic substack.

is injective with cokernel killed by F^h.

We will write $\mathcal{C}_{d,h}$ for the functor on $\mathrm{Nilp}^{\mathrm{ft}}$ sending A to the groupoid of φ-modules of F-height at most h over $\mathbb{A}_A^+$ that are moreover projective modules of rank d over $\mathbb{A}_A^+$, and $\mathcal{C}_{d,h}^a$ for its restriction to $\mathrm{Nilp}_a^{\mathrm{ft}}$. We have the following result of Pappas–Rapoport:

Theorem 2.1.2 (Theorem 4.1.6 of [2], originally Thm. 2.1 (a), Cor. 2.6 of [5])**.** *The stack $\mathcal{C}_{d,h}^a$ is an algebraic stack of finite presented over $\mathbb{Z}/p^a\mathbb{Z}$ with affine diagonal. Furthermore the morphism $\mathcal{C}_{d,h}^a \to \mathcal{R}_{d,\mathrm{free}}^a$ is representable by proper algebraic spaces of finite presentation.*

Remark 2.1.3. Let us explain why this morphism is proper: The moduli spaces of $\mathbb{A}_A^+$-lattices in $\mathbb{A}_A^d$ is given by an ind-proper ind-scheme (the affine Grassmannian for GL_d). The condition that $\mathscr{M}$ is $\varphi_{\mathscr{M}}$-stable together with the condition that $\varphi_{\mathscr{M}}$ is of F-height at most h give us upper and lower bounds for $\mathscr{M}$. So we should expect to land in a finite part of the affine Grassmannian. [The actual proof is not so simple, since we are not quite choosing lattices, but lattices up to σ-conjugation].

The way Emerton–Gee prove Theorem 2.0.1 is by showing that the morphism

$$\mathcal{C}_{d,h}^a \to \mathcal{R}_d^a$$

admits a 'scheme-theoretic image' which is itself an algebraic stack. They then show that

$$\varinjlim_h \mathcal{R}_{d,h}^a \to \mathcal{R}_d^a$$

is an isomorphism, which seems very plausible as it is sort of asserting the (local) existence of $\mathbb{A}_A^+$-lattices in φ-modules over A.

2.2. **A brief discussion of scheme-theoretic images of morphisms of (not necessarily algebraic) stacks.** Recall that if $f : X \to Y$ is a quasicompact morphism of schemes, then the scheme-theoretic image of f is well behaved. This is per definition the smallest closed subscheme $Z \subset Y$ such that $f : X \to Y$ factors through Z. When f is quasicompact then the resulting morphism $f : X \to Z$ is dominant, see Lemma 01R8 of the stacks project. When $f : \operatorname{Spec} B \to \operatorname{Spec} A$ corresponding to $f^\sharp : A \to B$ then the scheme-theoretic image of f is just just $\operatorname{Spec} A/\ker f^\sharp \subset \operatorname{Spec} A$. The formation of the scheme-theoretic image commutes with flat base change.

Similarly if $f : \mathcal{X} \to \mathcal{Y}$ is a quasicompact morphism of algebraic stacks, then the scheme-theoretic image of f can be defined as the smallest closed substack $\mathcal{Z} \subset \mathcal{Y}$ through which f factors [This always exists by Lemma 0CPU of the stacks project]. Moreover, the morphism $f : \mathcal{X} \to \mathcal{Z}$ has dense image on topological spaces, and the formation of scheme-theoretic images commutes with flat base change.

In the situation that we are in $f : \mathcal{X} \to \mathcal{Y}$ will be a quasi-compact morphism of stacks, where $\mathcal{X}$ is algebraic and locally of finite presentation, and where $\mathcal{Y}$ will be of finite presentation with representable diagonal of finite presentation. However $\mathcal{Y}$ will not necessarily be algebraic or even ind-algebraic. In this setting Emerton–Gee define a scheme-theoretic image $\mathcal{Z} \subset \mathcal{Y}$, which is just a substack. They then prove the following theorem

Theorem 2.2.1 (Theorem 1.1.1 of [3])**.** *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is proper. Suppose that $\mathcal{Y}$ admits (not necessarily Noetherian) versal rings at all finite type points, and that $\mathcal{Z}$ satisfies the Rim–Schlessinger conditions and admits effective Noetherian versal rings at all finite type points. Then $\mathcal{Z}$ is algebraic.*

Versal rings for $\mathcal{R}_{d,h}^a$ are given by unrestricted framed deformation rings for $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$, which are not necessarily Noetherian. It is explained in [1] that the versal rings for the scheme-theoretic image $\mathcal{R}_{d,h}^a$ these corresponds to the 'finite height' framed deformation rings of

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty})),$$

which were shown to be Noetherian in work of Kim [4]. The real difficulty then, is to prove effectivity, which is something we will not discuss here. The upshot of this is that we get the following result:

Theorem 2.2.2 (Corollary 4.2.3 of [2])**.** *The stack $\mathcal{R}_d$ is a limit preserving Ind-algebraic stack, whose diagonal is representable by algebraic spaces, affine and of finite presentation.*

3. Representability results for stacks of (φ, Γ)-modules

The main theorem that we are going to prove today is the following, where we let $\mathcal{X}_d^a$ be the restriction of $\mathcal{X}_d$ to $\mathrm{Nilp}_a^{\mathrm{ft}}$.

Theorem 3.0.1 (Proposition 3.4.12 of [1])**.** *The stack $\mathcal{X}_d^a$ can be written as a (countably indexed) inductive limit of algebraic spaces of finite presentation over $\mathbb{Z}/p^a\mathbb{Z}$, with transition maps given by closed immersions. Moreover the diagonal of $\mathcal{X}_d$ is representable by algebraic spaces, affine and of finite presentation.*

The difficulty in deducing this from Theorem 2.2.2 is with the continuity assumption on the Γ-action. Indeed, choose a topological generator $\gamma \in \Gamma$ and let $\Gamma_{\mathrm{disc}} \subset \Gamma$ be the subgroup it generates. Then we define an étale $(\varphi, \Gamma_{\mathrm{disc}})$-module over $A \in \mathrm{Nilp}^{\mathrm{ft}}$ to be an étale (φ)-module over A equipped with a semilinear action of Γ_{disc}. Let $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ be the stack on $\mathrm{Nilp}^{\mathrm{ft}}$ sending A to the groupoid of étale $(\varphi, \Gamma_{\mathrm{disc}})$-modules over A.

Let $\gamma_* : \mathcal{R}_d \to \mathcal{R}_d$ be the functor which takes an étale φ-module over A and pulls it back along $\gamma : \mathbb{A}_A \to \mathbb{A}_A$. It follows from the definitions of the stacky fiber product that the stack $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ can be identified with the fiber product

$$\begin{array}{ccc} \mathcal{R}_d^{\Gamma_{\mathrm{disc}}} & \longrightarrow & \mathcal{R}_d \\ \downarrow & & \downarrow \Gamma_\gamma \\ \mathcal{R}_d & \xrightarrow{\Delta} & \mathcal{R}_d \times \mathcal{R}_d \end{array}$$

where Δ is the diagonal morphism and Γ_γ is the graph of γ. Since the fiber product of algebraic stacks is again algebraic, the fiber product of (countably indexed) Ind-algebraic stacks is again a (countably indexed) Ind-algebraic stack.

There is a natural morphism $\mathcal{X}_d \to \mathcal{R}_d^{\Gamma}$ which takes an étale (φ, Γ)-module over A and sends it to the underlying étale $(\varphi, \Gamma_{\mathrm{disc}})$-module over A. Since Γ_{disc} is dense in Γ, this morphism is a monomorphism, i.e., fully faithful on A-points. Unfortunately, it isn't true that it is a closed (or open) substack, and therefore we are no closer to proving that $\mathcal{X}_d$ is an Ind-algebraic stack. The following lemma gives equivalent conditions for an étale $(\varphi, \Gamma_{\mathrm{disc}})$-module over A to arise from an étale (φ, Γ)-module over A.

Lemma 3.0.2. *The following are equivalent for an étale $(\varphi, \Gamma_{\mathrm{disc}})$-module over A, where A is a $\mathbb{Z}/p^a\mathbb{Z}$-algebra*

- *The action of Γ_{disc} extends to a continuous action of Γ.*
- *The action of γ on $M \otimes_{\mathbb{Z}/p^a\mathbb{Z}} \mathbb{F}_p$ is topologically nilpotent.*
- *There is a $\mathbb{A}_A^+$-lattice[2] $\mathscr{M} \subset M$ and $s \in \mathbb{Z}_{\geq 0}$ such that*

$$(\gamma^{p^s} - 1)\mathscr{M} \subset T\mathscr{M}.$$

- *For any $\mathbb{A}_A^+$-lattice $\mathscr{M} \subset M$ there is an $s \in \mathbb{Z}_{\geq 0}$ such that*

$$(\gamma^{p^s} - 1)\mathscr{M} \subset T\mathscr{M}.$$

It now follows that $\mathcal{X}_d$ is actually a stack in the fpqc topology, because the last condition can be checked fpqc locally. [At least, this is how I think of it, in [2] it is said that the fact that $\mathcal{X}_d$ is a stack is a consequence of the descent results of Drinfeld.]

3.1. **Weak Wach modules.** To prove our Ind-representability results for $\mathcal{X}_d$, we use a similar trick to the one in our proof of the Ind-representability of $\mathcal{R}_d$.

Definition 3.1.1. Let $A \in \mathrm{Nilp}^{\mathrm{ft}}$ and let $s, h \in \mathbb{Z}_{\geq 0}$. Then a *weak Wach-module* of height $\leq h$ and level $\leq s$ over A to be a projective rank d étale φ-module $\mathscr{M}$ of F-height at most h over $\mathbb{A}_A^+$ together with a semilinear action of Γ_{disc} on $M = \mathscr{M}[1/T]$ satisfying $(\gamma^{p^s} - 1)\mathscr{M} \subset T\mathscr{M}$. We will write $\mathcal{W}_{d,h,s}$ for the stack on $\mathrm{Nilp}^{\mathrm{ft}}$ sending A to the groupoid of weak Wach-modules of height $\leq h$ and level $\leq s$ over A.

[2] This means a finitely generated $\mathbb{A}_A^+$-submodule which generates M as an $\mathbb{A}_A$-module.

There is a natural monomorphism
$$\mathcal{W}_{d,h,s} \to \mathcal{R}_d^{\Gamma_{\text{disc}}} \times_{\mathcal{R}_d} \mathcal{C}_{d,h},$$
because the right hand side is the moduli stack of Weak Wach modules of height $\leq h$ (but with no condition on the level).

Proposition 3.1.2. *For each s this morphism is representable by a closed immersion of finite presentation. In particular for $s \geq s'$ the natural map*
$$\mathcal{W}_{d,h,s} \to \mathcal{W}_{d,h,s'}$$
are closed immersions of finite presentation.

The proof of this Proposition comes down to showing that the condition for the γ-action to satisfy
$$(\gamma^{p^s} - 1)\mathscr{M} \subset T\mathscr{M}$$
is cut out by finitely many equations, which happens in Proposition 3.3.5 of [1].

3.2. **Proof of Theorem 3.0.1.** We define $\mathcal{W}_{d,h} := \varinjlim_s \mathcal{W}_{d,h}$. This is an Ind-algebraic stack whose A-points can be identified with the groupoid of étale φ-modules $\mathscr{M}$ of F-height at most h over $\mathbb{A}_A^+$ together with a continuous semilinear action of Γ on $M = \mathscr{M}[1/T]$.

Note that there is a natural morphism $\mathcal{W}_{d,h} \to \mathcal{X}_d$ which just takes $\mathscr{M}$ and sends it to $M = \mathscr{M}[1/T]$ equipped with the Frobenius induced from $\mathscr{M}$ and its continuous Γ-action. Per definition there is a 2-fiber product diagram
$$\begin{array}{ccc} \mathcal{W}_{d,h} & \longrightarrow & \mathcal{R}_d^{\Gamma_{\text{disc}}} \times_{\mathcal{R}_d} \mathcal{C}_{d,h} \\ \downarrow & & \downarrow \\ \mathcal{X}_d & \longrightarrow & \mathcal{R}_d^{\Gamma_{\text{disc}}} \end{array}$$
Now if $h \leq h'$ there is a closed immersion $\mathcal{C}_{d,h} \to \mathcal{C}_{d,h'}$ compatible with the map to $\mathcal{R}_d$; this induces closed immersions
$$\mathcal{W}_{d,h} \to \mathcal{W}_{d,h'}$$
compatible with the map to $\mathcal{X}_d$. In particular there is an induced morphism
$$\varinjlim_h \mathcal{W}_{d,h} \to \mathcal{X}_d \to \mathcal{R}_d^{\Gamma_{\text{disc}}}.$$
We now define $\mathcal{X}_{d,h,s}^a$ to be the scheme-theoretic image of $\mathcal{W}_{d,h,s}^a \to \mathcal{R}_d^{\Gamma_{\text{disc}}}$. [Here we don't need any complicated definition of the scheme-theoretic image since the source is algebraic and the target is Ind-algebraic.] We now need to show that $\mathcal{X}_{d,h,s}^a$ is a closed substack of $\mathcal{X}_d^a$ rather than just of $\mathcal{R}_d^{\Gamma_{\text{disc}}}$. To be precise, we need to show that $\mathcal{X}_{d,h,s}^a$ is a substack of $\mathcal{X}_d^a$. This is more subtle than you might think, and is carried out in Lemma 3.4.9 of [1]. [By

definition of the scheme-theoretic image, we know that $\mathcal{X}^a_{d,h,s}(A) \subset \mathcal{X}^a_d(A)$ for Artin local A, so we need to show that a certain Γ_{disc}-action extends to a Γ-action on an étale (φ, Γ)-module over an object in Nilp^{ft} if it so extends for all Artin local quotients.]

Finally, we need to show that

$$\varinjlim_{h,s} \mathcal{X}^a_{d,h,s} = \mathcal{X}_d.$$

This last statement comes down to surjectivity, which just means that an étale (φ, Γ)-module M over A admits a $\mathbb{A}^+_A$-lattice with good properties, étale locally on A. This is straightforward to do when M is a free $\mathbb{A}_A$ module, and Emerton–Gee reduce to this case in their proof (see Proposition 3.4.10 of [1]).

References

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